C4 JAN 13

1. Given

$$f(x) = (2 + 3x)^{-3}, \qquad |x| < \frac{2}{3}$$

find the binomial expansion of f(x), in ascending powers of x, up to and including the term in x^3.

Give each coefficient as a simplified fraction.

(5)

$$2^{-3}\left(1+\frac{3}{2}x\right)^{-3} = \frac{1}{8}\left[1+(-3)\left(\frac{3}{2}x\right)+\frac{(-3)(-4)}{2}\left(\frac{3}{2}x\right)^2+\frac{(-3)(-4)(-5)}{6}\left(\frac{3}{2}x\right)^3\right]$$

$$= \frac{1}{8}\left[1-\frac{9}{2}x+\frac{27}{2}x^2-\frac{135}{4}x^3\right]$$

$$= \frac{1}{8}-\frac{9}{16}x+\frac{27}{16}x^2-\frac{135}{32}x^3$$

2. (a) Use integration to find

$$\int \frac{1}{x^3} \ln x \, dx$$

(5)

(b) Hence calculate

$$\int_1^2 \frac{1}{x^3} \ln x \, dx$$

(2)

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx \qquad u = \ln x \quad v' = x^{-3}$$

$$u' = \frac{1}{x} \quad v = -\frac{x^{-2}}{2}$$

$$= \frac{-1}{2x^2} \ln x + \int \frac{1}{2x^3} dx$$

$$= \frac{-1}{2x^2} \ln x - \frac{1}{4x^2} + c$$

b) $$\left[\frac{-1}{2x^2} \ln x - \frac{1}{4x^2}\right]_1^2 = \left(\frac{-1}{8} \ln 2 - \frac{1}{16}\right) - \left(-\frac{1}{4}\right)$$

$$= \frac{-1}{8} \ln 2 + \frac{3}{16} \qquad (\simeq 0.101 \text{ 3sf})$$

3. Express $\dfrac{9x^2 + 20x - 10}{(x + 2)(3x - 1)}$ in partial fractions.

(4)

$$\frac{9x^2+20x-10}{3x^2+5x-2}$$

$$9x^2+20x-10 = 3(3x^2+5x-2) + 5x-4$$

$$\frac{9x^2+20x-10}{(x+2)(3x-1)} = 3 + \frac{5x-4}{(x+2)(3x-1)}$$

$$5x-4 = A(3x-1) + B(x+2)$$

$x=-2 \quad -14 = -7A \quad \therefore A = 2$

$x = \frac{1}{3} \quad -\frac{7}{3} = \frac{7}{3}B \quad \therefore B = -1$

$$\therefore \quad 3 + \frac{2}{x+2} - \frac{1}{3x-1}$$

4.

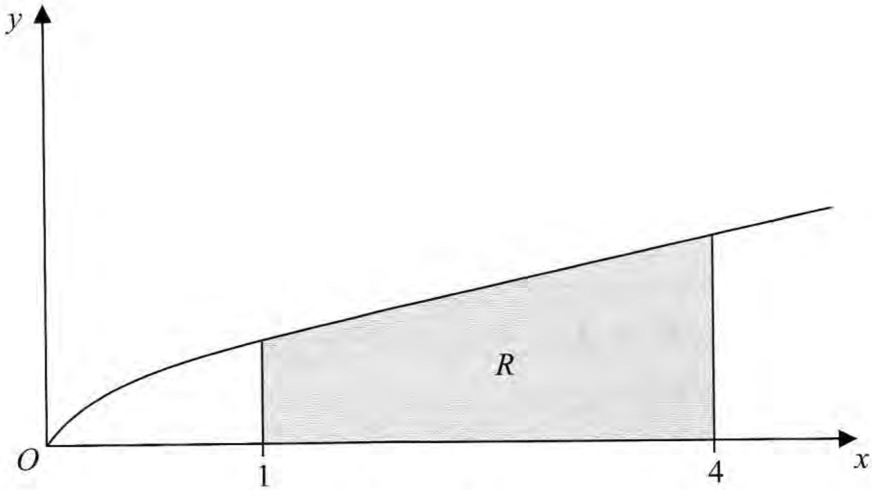


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \dfrac{x}{1+\sqrt{x}}$. The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis, the line with equation $x = 1$ and the line with equation $x = 4$.

(a) Complete the table with the value of y corresponding to $x = 3$, giving your answer to 4 decimal places.

(1)

x	1	2	3	4
y	0.5	0.8284		1.3333

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate of the area of the region R, giving your answer to 3 decimal places.

(3)

(c) Use the substitution $u = 1 + \sqrt{x}$, to find, by integrating, the exact area of R.

(8)

a) $3 \rightarrow 1.0981$

b) $h=1$ Area $\simeq \frac{1}{2}\left[0.5+1.3333+2(0.8284+1.0981)\right]$

$\simeq 2.843$ (3dp)

$$\int_1^4 \frac{x}{1+\sqrt{x}}\,dx$$

$$u = 1+\sqrt{x}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x}\,du$$

$x=4 \quad u=3 \qquad x=1 \quad u=2$

$$\sqrt{x} = u-1 \Rightarrow x=(u-1)^2$$

$$\therefore \int_2^3 \frac{(u-1)^2}{u}\,2(u-1)\,du = 2\int_1^3 \frac{u^3-3u^2+3u-1}{u}\,du$$

$$\Rightarrow 2\int_2^3 u^2-3u+3-\frac{1}{u}\,du = 2\left[\frac{1}{3}u^3-\frac{3}{2}u^2+3u-\ln u\right]_2^3$$

$$= 2\left[\left(\frac{9}{2}-\ln 3\right)-\left(\frac{8}{3}-\ln 2\right)\right] = 2\left(\frac{11}{6}+\ln 2-\ln 3\right)$$

$$= \frac{11}{3}+2\ln\frac{2}{3} \qquad \left[\text{or } \frac{11}{3}-\ln\frac{9}{4} \text{ etc}\right]$$

5.

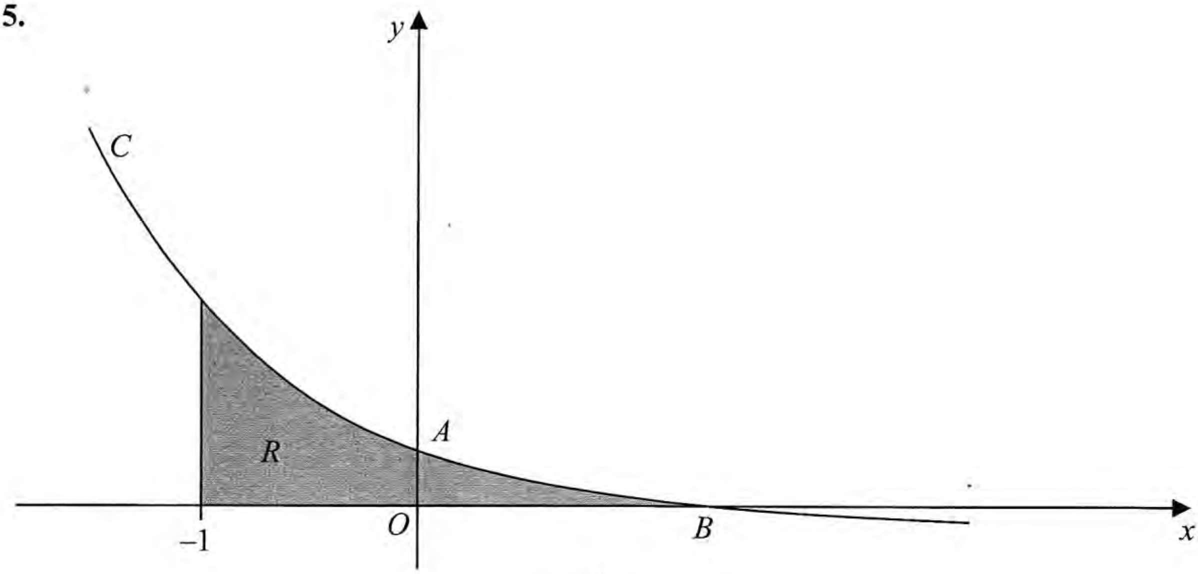


Figure 2

Figure 2 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \frac{1}{2}t, \quad y = 2^t - 1$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(a) Show that A has coordinates (0, 3).

(2)

(b) Find the x coordinate of the point B.

(2)

(c) Find an equation of the normal to C at the point A.

(5)

The region R, as shown shaded in Figure 2, is bounded by the curve C, the line $x = -1$ and the x-axis.

(d) Use integration to find the exact area of R.

(6)

a) $x = 0 \Rightarrow 1 = \frac{1}{2}t \Rightarrow t = 2 \quad \therefore \quad y = 2^2 - 1 = 3$ #

b) $y = 0 \Rightarrow 2^t = 1 \Rightarrow t = 0 \quad \Rightarrow \underline{x = 1}$

c) $\frac{dx}{dt} = -\frac{1}{2} \qquad \frac{dy}{dt} = 2^t \ln 2 \qquad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$\therefore \frac{dy}{dx} = -2^t \, 2\ln 2$.

Note $y = 2^t \Rightarrow \ln y = \ln 2^t \Rightarrow \ln y = t \times \ln 2$

$\Rightarrow \frac{d}{dt} \ln y = \frac{d}{dt}(t \times \ln 2) \Rightarrow \frac{1}{y}\frac{dy}{dt} = \ln 2$

$\Rightarrow \frac{dy}{dt} = y \ln 2 \quad \therefore \frac{dy}{dt} = 2^t \ln 2$ (not required if memorised)

at $x=0, t=2 \Rightarrow M_t = 8\ln 2 \Rightarrow M_n = \frac{-1}{8\ln 2}$

$A(0,3) \quad y - 3 = \frac{-1}{8\ln 2}(x-0) \Rightarrow y = \frac{-1}{8\ln 2}x + 3$

c) $x = 1 \Rightarrow t = 0 \quad x = -1 \Rightarrow -1 = 1 - \frac{t}{2} \quad \therefore t = 4$

$$\int_{x=-1}^{x=1} y\,dx = \int_{t=4}^{t=0} y \frac{dx}{dt}\,dt = \int_4^0 (2^t - 1) \times -\tfrac{1}{2}\,dt$$

$$= \frac{1}{2}\int_0^4 (2^t - 1)\,dt = \frac{1}{2}\left[\frac{2^t}{\ln 2} - t\right]_0^4 = \frac{1}{2}\left[\left(\frac{16}{\ln 2} - 4\right) - \frac{1}{\ln 2}\right]$$

$$= \frac{1}{2}\left(\frac{15}{\ln 2} - 4\right) = \frac{15}{2\ln 2} - 2$$

6.

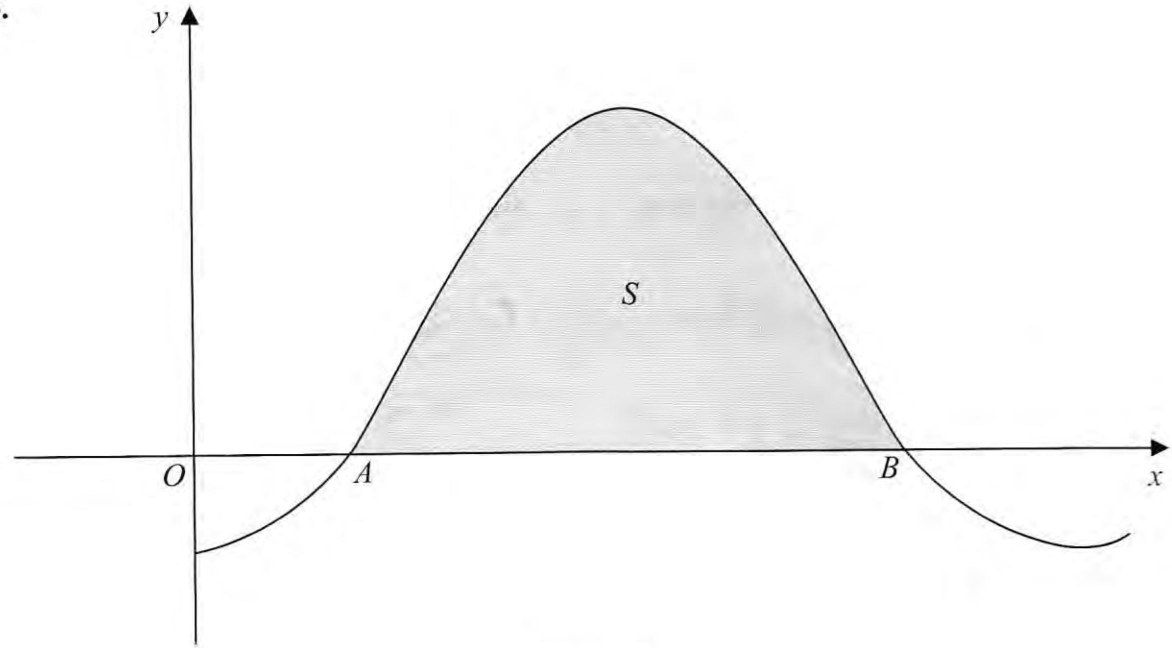


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = 1 - 2\cos x$, where x is measured in radians. The curve crosses the x-axis at the point A and at the point B.

(a) Find, in terms of π, the x coordinate of the point A and the x coordinate of the point B.
(3)

The finite region S enclosed by the curve and the x-axis is shown shaded in Figure 3. The region S is rotated through 2π radians about the x-axis.

(b) Find, by integration, the exact value of the volume of the solid generated.
(6)

$y=0 \Rightarrow 2\cos x = 1 \Rightarrow \cos x = \frac{1}{2}$

$\Rightarrow x = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \frac{5\pi}{3}$ $\quad A\left(\frac{\pi}{3}, 0\right)$ $B\left(\frac{5\pi}{3}, 0\right)$

b) Volume $= \pi \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-2\cos x)^2 \, dx$

$= \pi \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 4\cos^2 x - 4\cos x + 1 \, dx$

$$= \pi \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 4\left(\tfrac{1}{2}\cos 2x + \tfrac{1}{2}\right) - 4\cos x + 1 \; dx$$

$$= \pi \int_{\frac{\pi}{3}}^{\frac{5\pi}{2}} 2\cos 2x - 4\cos x + 3 \; dx$$

$$= \pi \Big[\sin 2x - 4\sin x + 3x \Big]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

$$= \pi \left[\left(-\frac{\sqrt{3}}{2} + 4\frac{\sqrt{3}}{2} + 5\pi \right) - \left(\frac{\sqrt{3}}{2} - 4\frac{\sqrt{3}}{2} + \pi \right) \right]$$

$$= \pi \left(3\sqrt{3} + 4\pi \right) \qquad \left[\pi 3\sqrt{3} + 4\pi^2 \right]$$

7. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \mathbf{r} = (9\mathbf{i} + 13\mathbf{j} - 3\mathbf{k}) + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

$$l_2 : \mathbf{r} = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

where λ and μ are scalar parameters.

(a) Given that l_1 and l_2 meet, find the position vector of their point of intersection.
(5)

(b) Find the acute angle between l_1 and l_2, giving your answer in degrees to 1 decimal place.
(3)

Given that the point A has position vector $4\mathbf{i} + 16\mathbf{j} - 3\mathbf{k}$ and that the point P lies on l_1 such that AP is perpendicular to l_1,

(c) find the exact coordinates of P.
(6)

$l_1 = \begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ $\quad$ $l_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ i) $9 + \lambda = 2 + 2\mu$

j) $13 + 4\lambda = -1 + \mu$

u) $-3 + -2\lambda = 1 + \mu$

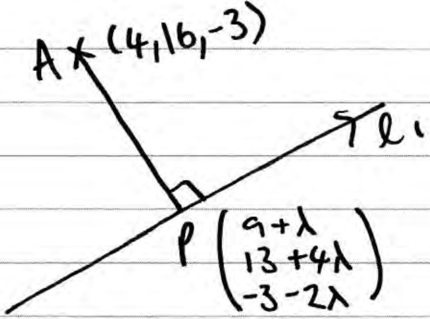
j) − u) $\Rightarrow 16 + 6\lambda = -2$

$6\lambda = -18$

$\lambda = -3$

$\therefore (6, 1, 3)$

b) $\cos\theta = \left| \dfrac{\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right|} \right| = \dfrac{4}{\sqrt{21}\sqrt{6}}$ $\quad \therefore \theta = 69.1°$



$$\vec{AP} = p - a \quad \begin{pmatrix} 9+\lambda \\ 13+4\lambda \\ -3-2\lambda \end{pmatrix} - \begin{pmatrix} 4 \\ 16 \\ -3 \end{pmatrix}$$

$$\vec{AP} = \begin{pmatrix} 5+\lambda \\ -3+4\lambda \\ -2\lambda \end{pmatrix}$$

$$\vec{AP} \cdot l_1 = 0 \Rightarrow \begin{pmatrix} 5+\lambda \\ -3+4\lambda \\ -2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = 0$$

$$\Rightarrow \quad 5 + \lambda - 12 + 16\lambda + 4\lambda = 0 \quad \Rightarrow \quad 21\lambda = 7 \therefore \lambda = \frac{1}{3}$$

$$\therefore P\left(\frac{28}{3}, \frac{43}{3}, \frac{-11}{3}\right).$$

8. A bottle of water is put into a refrigerator. The temperature inside the refrigerator remains constant at 3 °C and t minutes after the bottle is placed in the refrigerator the temperature of the water in the bottle is θ °C.

The rate of change of the temperature of the water in the bottle is modelled by the differential equation,

$$\frac{d\theta}{dt} = \frac{(3 - \theta)}{125}$$

(a) By solving the differential equation, show that,

$$\theta = Ae^{-0.008t} + 3$$

where A is a constant.

(4)

Given that the temperature of the water in the bottle when it was put in the refrigerator was 16 °C,

(b) find the time taken for the temperature of the water in the bottle to fall to 10 °C, giving your answer to the nearest minute.

(5)

$$\int \frac{1}{3-\theta}\, d\theta = \int \frac{1}{125}\, dt$$

$$-1\int \frac{-1}{3-\theta}\, d\theta = \frac{1}{125}t + c \Rightarrow -\ln|3-\theta| = 0.008t + c$$

$$\Rightarrow \ln|3-\theta| = -(0.008t + c) \Rightarrow 3-\theta = e^{-0.008t - c}$$

$$\Rightarrow 3-\theta = e^{-c} \times e^{-0.008t} \Rightarrow \theta = -e^{-c} \times e^{-0.008t} + 3$$

$$\therefore \theta = Ae^{-0.008t} + 3$$

b) $t = 0, \theta = 16 \Rightarrow 16 = A + 3 \Rightarrow A = 13.$

$$10 = 13e^{-0.008t} + 3 \Rightarrow 13e^{-0.008t} = 7 \Rightarrow e^{-0.008t} = \frac{7}{13}$$

$$\Rightarrow -\frac{1}{125}t = \ln\left(\frac{7}{13}\right) \Rightarrow t = -125\ln\left(\frac{7}{13}\right) = 77.37..$$

$\approx$ 77 min